Partially Invariant Solutions for Two-dimensional Ideal MHD Equations

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A generalization of the usual method of similarity analysis of differential equations, the method of partially invariant solutions, was introduced by Ovsiannikov. The degree of non-invariance of these solutions is characterized by the defect of invariance δ. We develop an algorithm leading to partially invariant solutions of quasilinear systems of first-order partial differential equations. We apply the algorithm to the non-linear equations of the two-dimensional non-stationary ideal MHD with a magnetic field perpendicular to the plane of motion.

Key words: Magnetohydrodynamics, Compressible flows, Group theory.

The usual method of similarity analysis of partial differential equations leading to group-invariant solutions is well described in the literature [1, 2]. A generalization of this method was introduced by soviet authors, especially Ovsiannikov [1]. It leads to solutions fulfilling a less strong invariance demand, called partially invariant solutions, which have a higher degree of freedom than invariant solutions. This may be important for adapting them to initial or boundary conditions. Apart from simple demonstration examples, Ovsiannikov's method has scarcely been applied to physical equations, perhaps as a result of its complexity. In this paper, we develop an algorithm applicable to quasilinear systems of first-order partial differential equations which yields two reduced systems leading to partially invariant solutions of the original system. Such solutions are characterized by the defect of invariance δ, which can be considered as their degree of non-invariance. For the simplest case, $\delta = 1$, we apply the algorithm to the two-dimensional non-stationary ideal MHD equations. These equations represent the isentropic motion of a two-dimensional plasma, if the magnetic field is perpendicular to the plane of motion and dissipative effects like thermal conductivity and electrical resistance are neglected. A detailed group analysis of this system has been done by Fuchs and Richter [3], who found its maximal symmetry group and a number of similarity solutions, as well as by Galas and Richter [4], who carried out the classification of its invariant solutions including the determination of optimal systems up to dimension 3. Every group-invariant solution is partially invariant with respect to any symmetry group which contains the regarded as a subgroup. In some cases, depending on the regarded symmetry group, the mentioned algorithm will yield only solutions which can also be found by the simpler method as invariant solutions. Therefore it seems desirable to recognize such cases as early as possible in order to exclude them from further computations. We show how this can be done by means of Ovsiannikov's reduction theorem. In this paper we assume that the reader is familiar with the basic tools of the similarity analysis.

1. The Concept of Partial Invariance

In what follows, we give a short review of the underlying mathematical concepts. For the details of the theory see [1]. We consider an r-dimensional transformation group G acting on $\mathbb{R}^N$ with the infinitesimal generators

$$X_i = \sum_{k=1}^{N} \alpha_i^k \frac{\partial}{\partial x^k}, \quad i = 1, \dots, r.$$

The number $r_* := \max \operatorname{rank}(\alpha_i^k)$ is called the *geometrical dimension* of G. A subset $M \subset \mathbb{R}^N$ given by m equations $\psi^i(x^1, \dots, x^N) = 0$, $i = 1, \dots, m$ with $m \leq N$ and rank $\partial(\psi^1, \dots, \psi^m)/\partial x = m$ for $x = (x^1, \dots, x^N) \in M$ is an $(N-m)$-dimensional differentiable manifold in $\mathbb{R}^N$. Its orbit under G is defined by

$$\mathcal{O}_G(M) := \{g(x) \mid x \in M, g \in G\}.$$

It can be shown that the inequality

$$\dim M \leqq \dim \mathcal{O}_G(M) \leqq \min\{r_* + \dim M, N\}$$

holds. By subtraction of $\dim M$ we can define the *defect of invariance* δ of M:

$$0 \leqq \delta := \dim \mathcal{O}_G(M) - \dim M \leqq \min\{r_*, m\}.$$

If $\delta = 0$, then $\mathcal{O}_G(M)$ is locally identical to M and M is called invariant. If $0 < \delta < \min\{r_*, m\}$, then M is called partially invariant with respect to G. For a given manifold M the defect can easily be calculated by the formula

$$\delta = \max_{x \in M} \operatorname{rank}(X_i \, \psi^j(x)).$$

Since $\mathcal{O}_G(M)$ is an invariant manifold, it can be represented by $m - \delta$ equations containing only invariants of G. Let t be the number of independent invariants of G. The dimension of $\mathcal{O}_G(M)$, projected to the space of invariants, is then given by $\varrho := t - m + \delta$. This number is called the *rank of invariance* of M with respect to G. The behaviour of a manifold M under a transformation group G is characterized by its rank and defect. If ϱ' and δ' denote the rank and defect of the same manifold M with respect to a subgroup $G' \subset G$, it can be shown that $\varrho' \geqq \varrho$ and $\delta' \leqq \delta$. If a subgroup $G' \subset G$ can be found in such a way that $\varrho' = \varrho$ and $\delta' = \delta$, the pair (M, G) is said to be reducible to (M, G'). Maximal reduction takes place if $\varrho' = \varrho$ and $\delta' = 0$, that is, M is invariant under the transformations of G'. A manifold M which is partially invariant under G but not invariant under some subgroup of G is called a *proper partially invariant* manifold. We now consider a system of first-order partial differential equations

$$F^i(x, u, u_x) = 0, \quad i = 1, \ldots, l. \tag{1}$$

Here and in what follows, we use the notation $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^m)$, $u_x = (u^1_{x^1}, \ldots, u^m_{x^n})$. The restriction to first-order equations is not essential in this paragraph. We presume that (1) fulfills the maximal rank condition $\operatorname{rank} \partial(F^1, \ldots, F^l)/\partial(x, u, u_x) = l$. Let a solution of this system be given by $u = \varphi(x)$, $u_x = \partial\varphi(x)/\partial x$. It is called a (partially) invariant solution with respect to G, where G has to be a symmetry group of (1), if its graph $\{(x, u) \mid u = \varphi(x)\} \subset \mathbb{R}^{n+m}$ is a (partially) invariant manifold in $\mathbb{R}^{n+m}$. We now assume the orbit of such a solution to be given by $m - \delta$ equations

$$\psi^i(x, u) = 0, \quad i = 1, \ldots, m - \delta$$

with

$$\operatorname{rank} \frac{\partial(\psi^1, \ldots, \psi^{m-\delta})}{\partial u} = m - \delta.$$

After an appropriate rearrangement of the components of u, this can be solved locally for some of them:

$$u^j = \tilde{\psi}^j(x, u^1, \ldots, u^\delta), \quad j = \delta + 1, \ldots, m.$$

Therefrom we get

$$u^j_{x^k} = \frac{\partial \tilde{\psi}^j}{\partial x^k} + \sum_{\nu=1}^{\delta} \frac{\partial \tilde{\psi}^j}{\partial u^\nu} u^\nu_{x^k}. \tag{2}$$

Now we can formulate Ovsiannikov's reduction theorem:

If the l equations (1) together with the $n(m - \delta)$ equations (2) can be solved for all first derivatives in such a way that

$$u^j_{x^k} = \Phi^j_k(x, u), \quad j = 1, \ldots, m, \quad k = 1, \ldots, n,$$

then every partially invariant solution lying in the regarded orbit is reducible to an invariant solution with respect to some subgroup of G. For the proof see [1].

2. Partially Invariant Solutions of Quasilinear Systems

In what follows, we consider a quasilinear system of first-order differential equations, that is a system of the form

$$A(x, u) u_x = B(x, u). \tag{3}$$

Here we use the matrix notation: $A(x, u)$ has to be read as an (l, nm)-matrix, u_x as an nm-column and $B(x, u)$ as an l-column. Suppose we are considering a symmetry group G of the geometrical dimension r_*. The number of independent invariants of G is then given by $t = n + m - r_*$. In the case of a solvable Lie algebra, a set of independent invariants can easily be found by means of a procedure which we want to skip here (see [2]). For the moment we denote them by $\lambda^1, \ldots, \lambda^t$. The orbit of a partially invariant solution forms an invariant manifold which by reason of the representation theorem [2] can be represented by $\mu := m - \delta$ equations containing only invariants:

$$\Psi^k(\lambda^1, \ldots, \lambda^t) = 0, \quad k = 1, \ldots, \mu$$

with

$$\operatorname{rank} \frac{\partial(\Psi^1, \ldots, \Psi^\mu)}{\partial(\lambda^1, \ldots, \lambda^t)} = \mu.$$

This can be solved for μ variables, e.g.

$$\lambda^i = \psi^i(\lambda^{\mu+1}, \dots, \lambda^t), \quad i = 1, \dots, \mu.$$

In oder to introduce $\lambda^1, \dots, \lambda^t$ as new dependent and independent variables into the system (3) we rename these variables as

$$\eta = (\eta^1, \dots, \eta^\mu) = (\lambda^1, \dots, \lambda^\mu) \quad \text{and}$$

$$\xi = (\xi^1, \dots, \xi^\varrho) = (\lambda^{\mu+1}, \dots, \lambda^t)$$

(note that $\mu + \varrho = t$). For their functional dependence on x and u we write $\xi = f^\xi(x, u)$ and $\eta = f^\eta(x, u)$. The orbit is now described by $\eta = \psi(\xi)$, but in the case of $\delta > 0$, we need further equations to specify the solutions. These, in general, will be non-invariant. For this purpose we preserve some of the old independent and dependent variables, the *parametric variables*, denoted by x^p and u^p, where $x^p \in \mathbb{R}^{n-\varrho}$ and $u^p \in \mathbb{R}^\delta$. The remaining are called the *main variables*: $x^h \in \mathbb{R}^\varrho$, $u^h \in \mathbb{R}^\mu$. Hence we have $x = (x^h, x^p)$, $u = (u^h, u^p)$. There is in general a certain freedom in choosing the parametric variables, but we require the following conditions to be fulfilled:

$$\text{rank} \frac{\partial f^\xi}{\partial x^h} = \varrho, \quad \text{rank} \frac{\partial f^\eta}{\partial u^h} = \mu, \quad \frac{\partial f^\xi}{\partial u^h} = 0. \quad (4)$$

Cases in which there is no possibility to satisfy (4) are excluded from further computations. As a result of (4) we can solve the system

$$\xi = f^\xi(x^h, x^p, u^p), \quad \eta = f^\eta(x^h, x^p, u^h, u^p)$$

for the main variables

$$x^h = f^{x^h}(\xi, x^p, u^p), \quad u^h = f^{u^h}(x^h, x^p, \eta, u^p).$$

Any solution of (3) can be written in the form

$$u = (u^h, u^p) = \varphi(x) = (\varphi^h(x), \varphi^p(x)).$$

For all solutions lying in the regarded orbit, we can evaluate the function φ^h and its derivatives

$$u^h = \varphi^h(x) = f^{u^h}(x, \psi(f^\xi(x, \varphi^p(x))), \varphi^p(x)),$$

$$u^h_x = \frac{\partial \varphi^h(x)}{\partial x}$$

$$= \frac{\partial f^{u^h}}{\partial x} + \frac{\partial f^{u^h}}{\partial \eta} \frac{\partial \psi}{\partial \xi} \left(\frac{\partial f^\xi}{\partial x} + \frac{\partial f^\xi}{\partial u^p} \frac{\partial \varphi^p}{\partial x} \right) + \frac{\partial f^{u^h}}{\partial u^p} \frac{\partial \varphi^p}{\partial x}.$$

This yields equations in the prolonged space of the new variables containing u^p and u^p_x as parameters:

$$u^h_x = \frac{\partial f^{u^h}}{\partial x} + \frac{\partial f^{u^h}}{\partial \eta} \eta_\xi \left(\frac{\partial f^\xi}{\partial x} + \frac{\partial f^\xi}{\partial u^p} u^p_x \right) + \frac{\partial f^{u^h}}{\partial u^p} u^p_x \quad (5)$$

(note that we use a symbolic notation omitting all signs of summation and indices). We now substitute this into the system of differential equations (3) and replace x^h and u^h by

$$x^h = f^{x^h}(\xi, x^p, u^p), \quad u^h = f^{u^h}(f^{x^h}(\xi, x^p, u^p), x^p, \eta, u^p).$$

The resulting system can be written as an inhomogeneous linear system of equations in u^p_x with coefficients depending on the new variables and the parametric variables:

$$L(\xi, \eta, \eta_\xi; x^p, u^p) u^p_x = P(\xi, \eta, \eta_\xi; x^p, u^p) \quad (6)$$

(read L as an $(l, n\delta)$-matrix, u^p_x as an $n\delta$-column and P as an l-column). This is called the *active system* because it can generate further equations as conditions for its solvability. At this point the reduction theorem is very useful: The active system is the composition of the equations (3) and (5), corresponding to (1) and (2). If $\text{rank}(L) = n\delta$, it is solvable for all components of u^p_x (and via (5) to all components of u^h_x) in the sense of linear algebra. If we assume that the orbit function ψ is known, we can substitute $\xi = f^\xi(x, u)$, $\eta = \psi(f^\xi(x, u))$,

$$\eta_\xi = \frac{\partial \psi}{\partial \xi} \bigg|_{\xi = f^\xi(x, u)}, \quad \text{and the requirements of the reduc-}$$

tion theorem are satisfied. This means that any solution which can be obtained from (6) will be an invariant solution with respect to some subgroup of G. Therefore we have $\text{rank}(L) < n\delta$ as a necessary condition for proper partially invariant solutions to exist.

3. The Reduced System

In general, the active system will be over-determined and there are certain conditions for its solvability. Here we have to distinguish between two kinds of conditions, which we call the *consistency conditions* and the *integrability conditions*. The former make sure the consistency of (6) in the sense of linear algebra (that is $\text{rank}(L) = \text{rank}(L, P)$), the latter guarantee the symmetry of the second derivatives u^p_{xx}. These conditions can be derived in an algorithmical way from the matrix (L, P). By elementary transformations of the rows and columns it is brought to the form

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \sigma_{1, q+1} & \dots & \sigma_{1, \delta n} & \tau_1 \\ 0 & & & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & 0 & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \sigma_{q, q+1} & \dots & \sigma_{q, \delta n} & \tau_q \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & \tau_{q+1} \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & \tau_l \end{bmatrix} \quad (7)$$

where the coefficients σ and τ depend on $(\xi, \eta, \eta_\xi; x^p, u^p)$. If $l > q = \mathrm{rank}\,(L)$, we get the consistency conditions directly from (7):

$$\tau_i(\xi, \eta, \eta_\xi; x^p, u^p) = 0 \quad \text{for} \quad i = q+1, \ldots, l.$$

It remains to find the integrability conditions. Because of the complexity of the general problem we confine ourselves to the case of $\delta = 1$, which is exclusively treated in the rest of this paper. Then u^p is a 1-dimensional variable and we can solve (6) with (7) for its derivatives

$$u^p_{x^i} = \tau_i(\xi, \eta, \eta_\xi; x^p, u^p) - \sum_{k=q+1}^{n} \sigma_{ik}(\xi, \eta, \eta_\xi; x^p, u^p)\, u^p_{x^k},$$
$$i = 1, \ldots, q. \quad (8)$$

From this we get

$$u^p_{x^i x^j} = \left(\frac{\partial \tau_i}{\partial \xi} + \frac{\partial \tau_i}{\partial \eta} \eta_\xi + \frac{\partial \tau_i}{\partial \eta_\xi} \eta_{\xi\xi} \right) \left(\frac{\partial f^\xi}{\partial x^j} + \frac{\partial f^\xi}{\partial u^p} u^p_{x^j} \right) + \frac{\partial \tau_i}{\partial x^j} + \frac{\partial \tau_i}{\partial u^p} u^p_{x^j}$$
$$- \sum_{k=q+1}^{n} \left[\left(\frac{\partial \sigma_{ik}}{\partial \xi} + \frac{\partial \sigma_{ik}}{\partial \eta} \eta_\xi + \frac{\partial \sigma_{ik}}{\partial \eta_\xi} \eta_{\xi\xi} \right) \left(\frac{\partial f^\xi}{\partial x^j} + \frac{\partial f^\xi}{\partial u^p} u^p_{x^j} \right) + \frac{\partial \sigma_{ik}}{\partial x^j} + \frac{\partial \sigma_{ik}}{\partial u^p} u^p_{x^j} \right] u^p_{x^k} - \sum_{k=q+1}^{n} \sigma_{ik} u^p_{x^k x^j}$$
$$(j = 1, \ldots, q).$$

We can evaluate $u^p_{x^k x^j}$:

$$u^p_{x^k x^j} = u^p_{x^j x^k} = \left(\frac{\partial \tau_j}{\partial \xi} + \frac{\partial \tau_j}{\partial \eta} \eta_\xi + \frac{\partial \tau_j}{\partial \eta_\xi} \eta_{\xi\xi} \right) \left(\frac{\partial f^\xi}{\partial x^k} + \frac{\partial f^\xi}{\partial u^p} u^p_{x^k} \right) + \frac{\partial \tau_j}{\partial x^k} + \frac{\partial \tau_j}{\partial u^p} u^p_{x^k}$$
$$- \sum_{\varkappa=q+1}^{n} \left[\left(\frac{\partial \sigma_{j\varkappa}}{\partial \xi} + \frac{\partial \sigma_{j\varkappa}}{\partial \eta} \eta_\xi + \frac{\partial \sigma_{j\varkappa}}{\partial \eta_\xi} \eta_{\xi\xi} \right) \left(\frac{\partial f^\xi}{\partial x^k} + \frac{\partial f^\xi}{\partial u^p} u^p_{x^k} \right) + \frac{\partial \sigma_{j\varkappa}}{\partial x^k} + \frac{\partial \sigma_{j\varkappa}}{\partial u^p} u^p_{x^k} \right] u^p_{x^\varkappa} - \sum_{\varkappa=q+1}^{n} \sigma_{j\varkappa} u^p_{x^\varkappa x^k}.$$

Analogously we get $u^p_{x^j x^i}$ and it follows

$$u^p_{x^i x^j} - u^p_{x^j x^i} = \Delta(\xi, \eta, \eta_\xi, \eta_{\xi\xi}, x^p, u^p, u^p_x) + \sum_{k=q+1}^{n} \sum_{\varkappa=q+1}^{n} \sigma_{ik}\, \sigma_{jk}\, u^p_{x^\varkappa x^k} - \sum_{k=q+1}^{n} \sum_{\varkappa=q+1}^{n} \sigma_{jk}\, \sigma_{i\varkappa}\, u^p_{x^\varkappa x^k},$$

where the terms with the second derivatives annulate one another. We substitute $u^p_{x^1}, \ldots, u^p_{x^q}$ by means of (8) and obtain integrability conditions of the form

$$\chi_j(\xi, \eta, \eta_\xi, \eta_{\xi\xi}; x^p, u^p, u^p_{x^{q+1}}, \ldots, u^p_{x^n}) = 0,$$
$$j = 1, \ldots, \frac{q(q-1)}{2}.$$

The consistency and integrability conditions are satisfied by every solution with its orbit given by $\eta = \psi(\xi)$. As G is a symmetry group, every such solution is transformed to another under the transformations of G. This means that the conditions themselves must be invariant under G or its first prolongation $\mathrm{pr}^{(1)} G$. So the consistency conditions, which do not depend on u^p_x, must be equivalent to conditions of the form $k_i(\xi, \eta, \eta_\xi) = 0$. This conclusion can not be drawn in the case of the integrability conditions because there

exist further invariants of $\mathrm{pr}^{(1)} G$ which depend on the derivative variables. Nevertheless we want to assume also the integrability conditions to be equivalent to conditions of the form $h_j(\xi, \eta, \eta_\xi, \eta_{\xi\xi}) = 0$. In practice this is true in most cases, and where it is not true we make it safe by additional conditions, being aware of the restriction of the set of solutions we get. The consistency and integrability conditions form a system of second-order differential equations in ξ and η, where $\xi \in \mathbb{R}^\varrho$, $\eta \in \mathbb{R}^\mu$. Since $\varrho < n$ and $\mu < m$ it is called the *reduced system* (or factor system). If $\varrho = 1$, one gets a system of ordinary differential equations and for $\varrho = 0$ the system is algebraic. The reduced system will, in general, be easier to solve than (3). Any solution

$\eta = \psi(\xi)$ of the reduced system describes the common orbit of some partially invariant solutions of (3). To find them we substitute $\eta = \psi(f^\xi(x, u^p))$ and

$$\eta_\xi = \left. \frac{\partial \psi(\xi)}{\partial \xi} \right|_{\xi = f^\xi(x, u^p)}$$

into the active system (6). In doing so it may be favourable to use the largely uncoupled form (7). The result is a self-consistent quasi-linear system of q partial differential equations for the dependent variables $u^p \in \mathbb{R}^\delta$, which we call the *remaining system*. In the case of $\delta = 1$, its general solution will contain a free function of $n - q$ variables. Any solution $u^p = \varphi^p(x)$ of the remaining system yields a partially invariant solution of (3) by the substitutions

$$u^h = f^{u^h}(x, \psi(f^\xi(x, \varphi^p(x))), \varphi^p(x)), \quad u^p = \varphi^p(x).$$

In many cases we can get further solutions from this by application of symmetry transformations which

are not contained in the normalizer of G. This procedure is analogous to that in the known case of invariant solutions, therefore we omit the details (see [2]).

4. Partially Invariant Solutions of the Ideal MHD Equations

We consider the system of the two-dimensional non-stationary ideal MHD equations:

$$\varrho(u_t + u u_x + v u_y) + R_H h h_x + p_x = 0,$$
$$\varrho(v_t + u v_x + v v_y) + R_H h h_y + p_y = 0,$$
$$\varrho_t + u \varrho_x + v \varrho_y + \varrho(u_x + v_y) = 0, \qquad (9)$$
$$h_t + u h_x + v h_y + h(u_x + v_y) = 0,$$
$$p_t + u p_x + v p_y + \gamma p(u_x + v_y) = 0,$$

where (x, y) are cartesian coordinates in the plane of motion, t is the time, (u, v) denote the velocity components, ϱ is the mass density (the double employment of the sign ϱ for the rank of invariance and the density should not lead to confusion), h is the value of the magnetic field which is assumed to be perpendicular to the (x, y)-plane, p is the gas pressure, γ the adiabatic exponent and R_H the magnetic pressure number. All variables are written in dimensionless quantities. This system contains 3 independent and 5 dependent variables, its maximal symmetry group $\hat{G}$ has been found by Fuchs and Richter [3]. In the case of $\gamma \neq 2$ the corresponding Lie algebra is of dimension 9, it is spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

$$X_5 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \quad X_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},$$

$$X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - 2\varrho\frac{\partial}{\partial \varrho},$$

$$X_8 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + 2\varrho\frac{\partial}{\partial \varrho},$$

$$X_9 = 2\varrho\frac{\partial}{\partial \varrho} + h\frac{\partial}{\partial h} + 2p\frac{\partial}{\partial p}.$$

In the special case of $\gamma = 2$, the Lie algebra becomes infinite-dimensional, but for our purpose it will be sufficient to consider one other generator:

$$X_* = \varrho\frac{\partial}{\partial h} - R_H \varrho h\frac{\partial}{\partial p}.$$

A great number of sub-algebras can be constructed by combination of these generators, but some of them are equivalent in a certain sense. By doing this systematically one is lead to complete lists of non-equivalent sub-algebras for each dimension, called optimal systems. These have been determined by Galas and Richter [4]. In what follows, we only consider some selected cases, confining ourselves to solutions of rank $\varrho = 1$ and defect $\delta = 1$, that is, the reduced systems consist of ordinary differential equations and the remaining systems contain only one single dependent variable. Since we have $n = 3$, $m = 5$ we get

$$\mu = m - \delta = 4, \quad t = \varrho + \mu = 5, \quad r_* = m + n - t = 3.$$

So we have to use subgroups of $\hat{G}$ of geometrical dimension 3. The optimal system offers a great variety of such subgroups, but most of them yield active systems with a coefficient matrix L of rank 3. As shown earlier, all solutions which can be derived from such an active system are reducible to invariant solutions, therefore we are looking for subgroups leading to rank$(L) < 3$. One may give a sufficient condition which can be used to recognize such subgroups by their Lie algebras in the special case of our system. We omit the details here, but we should remark that all the groups used for reduction of the system have been choosen by means of this criterion. Therefore it is not surprising that all active systems have the rank 2 or 1.

5. An Easy Example

We consider the group $G \subset \hat{G}$ with its Lie algebra spanned by the generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$$
$$X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - 2\varrho\frac{\partial}{\partial \varrho}.$$

Independent invariants of this group are t, v/u, $u^2\varrho$, h, p, these become the new variables $\xi, \eta^1, \ldots, \eta^4$. The only possible way to satisfy (4) is the choice $\xi = t$ and $x^h = t$. Morevoer we have to designate one parametric dependent variable out of u, v, ϱ (h and p are forbidden by (4)). As u and v are equivalent we set $u^p = u$, whereas the choice of ϱ yields more complicated equations in the active system. Written in the above mentioned notation we have

$$\xi = f^\xi(x, y, t; u, v, \varrho, h, p) = t,$$
$$\eta^1 = f^{\eta^1}(x, y, t; u, v, \varrho, h, p) = \frac{v}{u},$$

$$\eta^2 = f^{\eta^2}(x, y, t; u, v, \varrho, h, p) = u^2 \varrho,$$

$$\eta^3 = f^{\eta^3}(x, y, t; u, v, \varrho, h, p) = h,$$

$$\eta^4 = f^{\eta^4}(x, y, t; u, v, \varrho, h, p) = p,$$

$$t = f^t(x, y, \xi; \eta^1, \ldots, \eta^4; u) = \xi,$$

$$v = f^v(x, y, t; \eta^1, \ldots, \eta^4; u) = \eta^1 u,$$

$$\varrho = f^\varrho(x, y, t; \eta^1, \ldots, \eta^4; u) = \frac{\eta^2}{u^2},$$

$$h = f^h(x, y, t; \eta^1, \ldots, \eta^4; u) = \eta^3,$$

$$p = f^p(x, y, t; \eta^1, \ldots, \eta^4; u) = \eta^4.$$

By substituting the main variables v, ϱ, h, p and their derivatives according to (5) in (9), we get the active system

$$\frac{\eta^2}{u^2}(u_t + u u_x + \eta^1 u u_y) = 0,$$

$$\frac{\eta^2}{u^2}(\eta^1 u_t + (\eta^1)' u + \eta^1 u u_x + (\eta^1)^2 u u_y) = 0,$$

$$-\frac{2\eta^2}{u^3} u_t + \frac{(\eta^2)'}{u^2} - \frac{\eta^2}{u^2} u_x - \frac{\eta^1 \eta^2}{u^2} u_y = 0,$$

$$(\eta^3)' + \eta^3(u_x + \eta^1 u_y) = 0,$$

$$(\eta^4)' + \gamma \eta^4(u_x + \eta^1 u_y) = 0,$$

or in matrix form (6)

$$\underbrace{\begin{pmatrix} u & \eta^1 u & 1 \\ \eta^1 u & (\eta^1)^2 u & \eta^1 \\ -\dfrac{\eta^2}{u^2} & -\dfrac{\eta^1 \eta^2}{u^2} & -\dfrac{2\eta^2}{u^3} \\ \eta^3 & \eta^1 \eta^3 & 0 \\ \gamma \eta^4 & \gamma \eta^1 \eta^4 & 0 \end{pmatrix}}_{L} \begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ -(\eta^1)' u \\ -\dfrac{(\eta^2)'}{u^2} \\ -(\eta^3)' \\ -(\eta^4)' \end{pmatrix}}_{P}.$$

Elementary transformations yield

$$\begin{pmatrix} 1 & 0 & \eta^1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_t \\ u_y \end{pmatrix} = \begin{pmatrix} -\dfrac{(\eta^2)'}{\eta^2} \\ \dfrac{(\eta^2)'}{\eta^2} u \\ (\eta^1)' \\ \eta^2(\eta^3)' - \eta^3(\eta^2)' \\ \eta^2(\eta^4)' - \gamma \eta^4(\eta^2)' \end{pmatrix} \quad (10)$$

(this is only correct for $u \not\equiv 0$ and $\eta^2 \not\equiv 0$, but the latter is physically trivial because it is equivalent to a not identically vanishing mass density).

At this stage we can formulate the consistency conditions immediately:

$$(\eta^1)' = 0, \quad \eta^2(\eta^3)' - \eta^3(\eta^2)' = 0,$$

$$\eta^2(\eta^4)' - \gamma \eta^4(\eta^2)' = 0. \quad (11)$$

Moreover we obtain

$$u_x = -\frac{(\eta^2)'}{\eta^2} - \eta^1 u_y, \quad u_t = \frac{(\eta^2)'}{\eta^2} u$$

and can derive the integrability condition from

$$u_{xt} = -\frac{d}{d\xi}\left(\frac{(\eta^2)'}{\eta^2}\right) - (\eta^1)' u_y - \eta^1 u_{yt},$$

$$u_{yt} = u_{ty} = \frac{(\eta^2)'}{\eta^2} u_y,$$

$$u_{tx} = \frac{(\eta^2)'}{\eta^2} u_x.$$

We find

$$u_{xt} - u_{tx} = -\frac{d}{d\xi}\left(\frac{(\eta^2)'}{\eta^2}\right) - (\eta^1)' u_y - \eta^1 \frac{(\eta^2)'}{\eta^2} u_y$$

$$+ \left(\frac{(\eta^2)'}{\eta^2}\right)^2 + \eta^1 \frac{(\eta^2)'}{\eta^2} u_y$$

$$= -\frac{d}{d\xi}\left(\frac{(\eta^2)'}{\eta^2}\right) + \left(\frac{(\eta^2)'}{\eta^2}\right)^2 - (\eta^1)' u_y,$$

so that with (11), the integrability condition is equivalent to

$$\frac{d}{d\xi}\left(\frac{(\eta^2)'}{\eta^2}\right) - \left(\frac{(\eta^2)'}{\eta^2}\right)^2 = 0.$$

This differential equation of second order and the consistency conditions (11) form the reduced system. It can easily be integrated by elementary methods and its general solution is

$$\eta^1 = C_1, \quad \eta^2 = \frac{C_2}{\xi + C_5}, \quad \eta^3 = \frac{C_3}{\xi + C_5}, \quad \eta^4 = \left(\frac{C_4}{\xi + C_5}\right)^\gamma.$$

Substituting this into the active system (10) and with the re-substitution $\xi = t$, we get the remaining system for u:

$$u_x + C_1 u_y = \frac{1}{t + C_5}, \quad u_t = -\frac{u}{t + C_5}.$$

Solving the first equation by the method of characteristics yields

$$u = \frac{x}{t + C_5} + f(y - C_1 x, t)$$

with an arbitrary differentiable two-argument function f. By substituting this into the second equation we get a differential equation for f:

$$f_t = -\frac{f}{t+C_5}.$$

Its solution is

$$f(y-C_1 x, t) = \frac{g(y-C_1 x)}{t+C_5}$$

with an arbitrary differentiable function g. By re-substitution we get a set of solutions of (9):

$$u = \frac{x+g(y-C_1 x)}{t+C_5}, \qquad v = C_1 \frac{x+g(y-C_1 x)}{t+C_5},$$

$$\varrho = \frac{C_2(t+C_5)}{(x+g(y-C_1 x))^2}, \quad h = \frac{C_3}{t+C_5}, \quad p = \left(\frac{C_4}{t+C_5}\right)^\gamma.$$

These solutions are partially invariant under the group G. It can be shown that they are only reducible to invariant solutions with respect to some subgroup of G if the functions F_1, F_2, F_3 with

$$F_1(z) = 1 - C_1 g'(z), \quad F_2(z) = g'(z),$$

$$F_3(z) = z g'(z) - g(z)$$

are linearly dependent, that is if g is constant or linear. Finally, we can get further solutions of (9) by applying symmetry transformations which are not contained in the normalizer of G. In our case, these are the global transformations according to the generators X_4 and X_5. Omitting the details we give only the result:

$$u = \frac{x-\varepsilon^4 t + g(y-\varepsilon^5 t - C_1(x-\varepsilon^4 t))}{t+C_5} + \varepsilon^4,$$

$$v = C_1 \cdot \frac{x-\varepsilon^4 t + g(y-\varepsilon^5 t - C_1(x-\varepsilon^4 t))}{t+C_5} + \varepsilon^5,$$

$$\varrho = C_2 \cdot \frac{t+C_5}{(x-\varepsilon^4 t + g(y-\varepsilon^5 t - C_1(x-\varepsilon^4 t)))^2},$$

$$h = \frac{C_3}{t+C_5},$$

$$p = \left(\frac{C_4}{t+C_5}\right)^\gamma,$$

where ε^4 and ε^5 can be regarded as parameters of the solution. These solutions are partially invariant under certain subgroups of $\hat{G}$ different from G for ε^4, $\varepsilon^5 \neq 0$.

6. Further Examples

In what follows, we list four other sets of partially invariant solutions of (9) together with the corresponding group invariants, active systems, reduced systems and remaining systems. The general computing algorithm is the same as described in the previous paragraph, but there arise some particularities which we point out in each case.

a) We consider the Lie algebra with the basis $\{X_1, X_4, X_5\}$. Independent invariants are

$$\xi = t, \quad \eta^1 = vt - y, \quad \eta^2 = \varrho, \quad \eta^3 = h, \quad \eta^4 = p.$$

We choose $u^p = u$, $x^h = t$. The active system is

$$\begin{pmatrix} \eta^2 u & \frac{1}{t}\eta^2(\eta^1+y) & \eta^2 \\ 0 & 0 & 0 \\ \eta^2 & 0 & 0 \\ \eta^3 & 0 & 0 \\ \gamma\eta^4 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{t}\eta^2(\eta^1)' \\ -\left((\eta^2)' + \frac{\eta^2}{t}\right) \\ -\left((\eta^3)' + \frac{\eta^3}{t}\right) \\ -\left((\eta^4)' + \gamma\frac{\eta^4}{t}\right) \end{pmatrix}$$

and we have rank $(L) = 2$. For $u \not\equiv 0$, $\eta^2 \not\equiv 0$ the reduced system is

$$(\eta^1)' = 0, \quad \eta^2(\eta^3)' - \eta^3(\eta^2)' = 0, \quad \eta^2(\eta^4)' - \gamma\eta^4(\eta^2)' = 0,$$

$$\frac{d}{d\xi}\left(\frac{(\eta^2)'}{\eta^2} + \frac{1}{\xi}\right) - \left(\frac{(\eta^2)'}{\eta^2} + \frac{1}{\xi}\right)^2 = 0$$

with the general solution

$$\eta^1 = C_1, \qquad \eta^2 = \frac{C_2}{\xi(\xi+C_5)},$$

$$\eta^3 = \frac{C_3}{\xi(\xi+C_5)}, \quad \eta^4 = \left(\frac{C_4}{\xi(\xi+C_5)}\right)^\gamma.$$

The remaining system is

$$u_x = \frac{1}{t+C_5}, \quad (y+C_1)u_y + tu_t = -\frac{t}{t+C_5}u$$

with the general solution

$$u = \frac{x+g\left(\frac{y+C_1}{t}\right)}{t+C_5},$$

where g denotes an arbitrary differentiable function. After appropriate symmetry transformations we ob-

tain the following set of solutions of (9):

$$u = \frac{x \cos \varepsilon^6 - y \sin \varepsilon^6 + g\left(\dfrac{x \sin \varepsilon^6 + y \cos \varepsilon^6 + C_1}{t - \varepsilon^3}\right)}{t - \varepsilon^3 + C_5} \cos \varepsilon^6 + \frac{x \sin \varepsilon^6 + y \cos \varepsilon^6 + C_1}{t - \varepsilon^3} \sin \varepsilon^6,$$

$$v = - \frac{x \cos \varepsilon^6 - y \sin \varepsilon^6 + g\left(\dfrac{x \sin \varepsilon^6 + y \cos \varepsilon^6 + C_1}{t - \varepsilon^3}\right)}{t - \varepsilon^3 + C_5} \sin \varepsilon^6 + \frac{x \sin \varepsilon^6 + y \cos \varepsilon^6 + C_1}{t - \varepsilon^3} \cos \varepsilon^6,$$

$$\varrho = \frac{C_2}{(t - \varepsilon^3)(t - \varepsilon^3 + C_5)}, \quad h = \frac{C_4}{(t - \varepsilon^3)(t + C_5)}, \quad p = \left(\frac{C_4}{(t - \varepsilon^3)(t + C_5)}\right)^\gamma.$$

Although we had found $\text{rank}(L) = 2$, for $\varepsilon^3 = \varepsilon^6 = 0$ these solutions are invariant under the subgroup of G with the generator $X = -C_5 X_1 + X_4$. This shows that $\text{rank}(L) < 3$ is necessary but not sufficient for the solutions to be proper partially invariant.

b) We consider the Lie algebra with the basis $\{X_4, X_5, X_6\}$. Independent invariants are

$$\xi = t, \quad \eta^1 = (ut - x)^2 + (vt - y)^2, \quad \eta^2 = \varrho, \quad \eta^3 = h, \quad \eta^4 = p.$$

We choose $u^p = u$, $x^h = t$. The active system is

$$
\begin{bmatrix}
\eta^2 u & \dfrac{\eta^2(\sqrt{\eta^1 - (ut-x)^2} + y)}{t} & \eta^2 \\[3mm]
-\dfrac{\eta^2 u(ut-x)}{\sqrt{\eta^1 - (ut-x)^2}} & -\dfrac{\eta^2(\sqrt{\eta^1 - (ut-x)^2} + y)(ut-x)}{t\sqrt{\eta^1 - (ut-x)^2}} & -\dfrac{\eta^2(ut-x)}{\sqrt{\eta^1 - (ut-x)^2}} \\[3mm]
\eta^2 & -\dfrac{\eta^2(ut-x)}{\sqrt{\eta^1 - (ut-x)^2}} & 0 \\[3mm]
\eta^3 & -\dfrac{\eta^3(ut-x)}{\sqrt{\eta^1 - (ut-x)^2}} & 0 \\[3mm]
\gamma \eta^4 & -\gamma \dfrac{\eta^4(ut-x)}{\sqrt{\eta^1 - (ut-x)^2}} & 0
\end{bmatrix}
\begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix} =
\begin{bmatrix}
0 \\[3mm]
-\dfrac{\eta^2(\eta^1)'}{2t\sqrt{\eta^1 - (ut-x)^2}} \\[3mm]
-(\eta^2)' - \dfrac{\eta^2}{t} \\[3mm]
-(\eta^3)' - \dfrac{\eta^3}{t} \\[3mm]
-(\eta^4)' - \gamma \dfrac{\eta^4}{t}
\end{bmatrix}
$$

and we have $\text{rank}(L) = 2$. For $\eta^1 - (ut-x)^2 \neq 0$, $\eta^2 \neq 0$ the reduced system is

$$(\eta^1)' = 0, \quad \eta^2(\eta^3)' - \eta^3(\eta^2)' = 0, \quad \eta^2(\eta^4)' - \gamma \eta^4(\eta^2)' = 0, \quad \frac{d}{d\xi}\left(\frac{(\eta^2)'}{\eta^2} + \frac{1}{\xi}\right) - \left(\frac{(\eta^2)'}{\eta^2} + \frac{1}{\xi}\right)^2 = 0$$

with the general solution

$$\eta^1 = C_1, \quad \eta^2 = \frac{C_2}{\xi(\xi + C_5)}, \quad \eta^3 = \frac{C_3}{\xi(\xi + C_5)}, \quad \eta^4 = \left(\frac{C_4}{\xi(\xi + C_5)}\right)^\gamma.$$

The remaining system is

$$u_x - \frac{ut - x}{\sqrt{C_1 - (ut-x)^2}} u_y = \frac{1}{t + C_5}, \quad u_t + \left(\frac{y}{t} + \frac{C_1 + x(ut-x)}{t\sqrt{C_1 - (ut-x)^2}}\right) u_y = -\frac{u}{t + C_5}.$$

Its general solution can only be given in the implicit form

$$C_5 \frac{y}{t} + \frac{t + C_5}{t}\sqrt{C_1 - (ut-x)^2} - g(u(t + C_5) - x) = 0$$

with an arbitrary differentiable function g. After appropriate symmetry transformations we obtain the following set of implicit solutions of (9):

$$C_5 \frac{y - \varepsilon^2}{t - \varepsilon^3} + \frac{t - \varepsilon^3 + C_5}{t - \varepsilon^3} \sqrt{C_1 - (u(t - \varepsilon^3) - x + \varepsilon^1)^2} - g(u(t - \varepsilon^3 + C_5) - x + \varepsilon^1) = 0,$$

$$v = \frac{y - \varepsilon^2 + \sqrt{C_1 - (u(t - \varepsilon^3) - x + \varepsilon^1)^2}}{t - \varepsilon^3}, \quad \varrho = \frac{C_2}{(t - \varepsilon^3)(t - \varepsilon^3 + C_5)},$$

$$h = \frac{C_3}{(t - \varepsilon^3)(t - \varepsilon^3 + C_5)}, \quad p = \left(\frac{C_4}{(t - \varepsilon^3)(t - \varepsilon^3 + C_5)} \right)^\gamma.$$

For $\varepsilon^1 = \varepsilon^2 = \varepsilon^3 = 0$ these solutions are proper partially invariant if the expressions 1, $g'(z)$, $g(z) g'(z) + z$ with $z = u(t + C_5) - x$ are linearly independent.

c) We consider the Lie algebra with the basis $\{X_1, X_5 + a X_4, X_7\}$. Independent invariants are

$$\xi = t, \quad \eta^1 = \frac{vt - y}{ut - ay}, \quad \eta^2 = (ut - ay)^2 \varrho, \quad \eta^3 = h, \quad \eta^4 = p.$$

We choose $u^p = u$, $x^h = t$. The active system is

$$\begin{bmatrix} \dfrac{\eta^2 u}{(ut - ay)^2} & \dfrac{\eta^1 \eta^2}{t(ux - ay)} + \dfrac{\eta^2 y}{t(ut - ay)^2} & \dfrac{\eta^2}{(ut - ay)^2} \\[2mm] \dfrac{\eta^1 \eta^2 u}{(ut - ay)^2} & \dfrac{(\eta^1)^2 \eta^2}{t(ut - ay)} + \dfrac{\eta^1 \eta^2 y}{t(ut - ay)^2} & \dfrac{\eta^1 \eta^2}{(ut - ay)^2} \\[2mm] -\dfrac{\eta^2(ut + ay)}{(ut - ay)^3} & -\dfrac{\eta^1 \eta^2}{(ut - ay)^2} - \dfrac{2\eta^2 y}{(ut - ay)^3} & -\dfrac{2\eta^2 t}{(ut - ay)^3} \\[2mm] \eta^3 & \eta^1 \eta^3 & 0 \\[2mm] \gamma \eta^4 & \gamma \eta^1 \eta^4 & 0 \end{bmatrix} \begin{pmatrix} u_x \\ u_y \\ u_t \end{pmatrix} = \begin{bmatrix} 0 \\[2mm] -\dfrac{(\eta^1)' \eta^2}{t(ut - ay)} + \dfrac{\eta^1 \eta^2 (a\eta^1 - 1)}{t^2(ut - ay)} \\[2mm] -\dfrac{\eta^2}{(ut - ay)^2} - \dfrac{\eta^2(a\eta^1 - 1)}{t(ut - ay)^2} \\[2mm] -(\eta^3)' + \dfrac{\eta^3(a\eta^1 - 1)}{t} \\[2mm] -(\eta^4)' + \gamma \dfrac{\eta^4(a\eta^1 - 1)}{t} \end{bmatrix}$$

and we have rank$(L) = 2$. For $ut - ay \not\equiv 0$, $\eta^2 \not\equiv 0$, $\eta^3 \not\equiv 0$ the reduced system is

$$(\eta^1)' - \eta^1 \frac{a\eta^1 - 1}{\xi} = 0, \quad \eta^2(\eta^3)' - \eta^3(\eta^2)' + \frac{2}{\xi} \eta^2 \eta^3(a\eta^1 - 1) = 0, \quad \eta^3(\eta^4)' - \gamma \eta^4(\eta^3)' = 0,$$

$$\frac{d}{d\xi} \left(\frac{(\eta^3)'}{\eta^3} - \frac{a\eta^1 - 1}{\xi} \right) - \left(\frac{(\eta^3)'}{\eta^3} - \frac{a\eta^1 - 1}{\xi} \right)^2 + \left((\eta^1)' + \eta^1 \frac{a\eta^1 - 1}{\xi} \right) u_y = 0.$$

Here we have an example for the case that the integrability condition depends on the derivative variable u_y. In order to eliminate this difficulty we require additionally

$$(\eta^1)' + \eta^1 \frac{a\eta^1 - 1}{\xi} = 0.$$

Together with the first consistency condition this yields two cases:

(i) $\eta^1 \equiv \dfrac{1}{a}$ for $a \neq 0$, (ii) $\eta^1 \equiv 0$ for $a \in \mathbb{R}$.

The general solution of the reduced system is in case (i)

$$\eta^1 = \frac{1}{a}, \quad \eta^2 = \frac{C_2}{\xi + C_5}, \quad \eta^3 = \frac{C_3}{\xi + C_5}, \quad \eta^4 = \left(\frac{C_4}{\xi + C_5} \right)^\gamma,$$

and the remaining system is

$$u_x + \frac{1}{a} u_y = \frac{1}{t + C_5}, \quad u_t = -\frac{u}{t + C_5}$$

with the general solution

$$u = \frac{x + g(x - ay)}{t + C_5},$$

where g denotes an arbitrary differentiable function. Moreover we obtain

$$v = \frac{x + g(x - a y)}{a(t + C_5)}, \quad \varrho = \frac{C_2(t + C_5)}{(x t + g(x - a y) t - a y(t + C_5))^2},$$

$$h = \frac{C_3}{t + C_5}, \quad p = \left(\frac{C_4}{t + C_5}\right)^\gamma.$$

These solutions are proper partially invariant if the expressions $1, 1 + g'(z), z g'(z) - g(z)$ with $z = x - a y$ are linearly independent.

The general solution of the reduced system is in case (ii)

$$\eta^1 = 0, \quad \eta^2 = \frac{C_2 \xi}{\xi + C_5},$$

$$\eta^3 = \frac{C_3}{\xi(\xi + C_5)}, \quad \eta^4 = \left(\frac{C_4}{\xi(\xi + C_5)}\right)^\gamma,$$

and the remaining system is

$$u_x = \frac{1}{t + C_5}, \quad u_t + \frac{y}{t} u_y = -\frac{u}{t + C_5}$$

with the general solution

$$u = \frac{x + g\left(\frac{y}{t}\right)}{t + C_5},$$

where g denotes an arbitrary differentiable function. Moreover we get

$$v = \frac{y}{t}, \quad \varrho = \frac{C_2 t(t + C_5)}{\left(x t + g\left(\frac{y}{t}\right) t - a y(t + C_5)\right)^2},$$

$$h = \frac{C_3}{t(t + C_5)}, \quad p = \left(\frac{C_4}{t(t + C_5)}\right)^\gamma.$$

These solutions are proper partially invariant if the expressions $1, g'(z) - a C_5, z g'(z) - g(z)$ with $z = y/t$ are linearly independent.

In both cases we can obtain further solutions by the following symmetry transformations:

$$x \rightarrow x \cos \varepsilon^6 - y \sin \varepsilon^6 - \varepsilon^4 t,$$

$$y \rightarrow x \sin \varepsilon^6 + y \cos \varepsilon^6 - \varepsilon^5 t - \varepsilon^2,$$

$$t \rightarrow t - \varepsilon^3,$$

$$u \leftarrow (u + \varepsilon^4) \cos \varepsilon^6 + (v + \varepsilon^5) \sin \varepsilon^6,$$

$$v \leftarrow -(u + \varepsilon^4) \sin \varepsilon^6 + (v + \varepsilon^5) \cos \varepsilon^6.$$

We don't give here the expressions of the obtained solutions by reason of their length.

d) Finally, we consider the Lie algebra with the basis $\{X_1, X_7 + 2 X_9, X_*\}$ for the special case of $\gamma = 2$. Independent invariants are

$$\xi = t, \quad \eta^1 = \frac{u}{y}, \quad \eta^2 = \frac{v}{y}, \quad \eta^3 = \frac{\varrho}{y^2}, \quad \eta^4 = \frac{\frac{R_H}{2} h^2 + p}{y^4}.$$

We choose $u^p = h$, $x^h = t$. The active system is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta^1 y & \eta^2 y & 1 \\ -\eta^1 y R_H h & -\eta^2 y R_H h & -R_H h \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_t \end{pmatrix}$$

$$= \begin{pmatrix} y^3 \eta^3((\eta^1)' + \eta^1 \eta^2) \\ y^3(\eta^3((\eta^2)' + (\eta^2)^2) + 4\eta^4) \\ y^2((\eta^3)' + 3\eta^2 \eta^3) \\ -h \eta^2 \\ -y^4((\eta^4)' + 6\eta^2 \eta^4) + R_H h^2 \eta^2 \end{pmatrix}.$$

Here we have rank$(L) = 1$. Therefore we obtain four consistency conditions but no integrability condition. For $\eta^3 \not\equiv 0$ the reduced system is

$$(\eta^1)' + \eta^1 \eta^2 = 0, \quad \eta^3((\eta^2)' + (\eta^2)^2) + 4\eta^4 = 0,$$

$$(\eta^3)' + 3\eta^2 \eta^3 = 0, \quad (\eta^4)' + 6\eta^2 \eta^4 = 0.$$

A set of special solutions of this system is

$$\eta^1 = \frac{C_1}{\xi^{2/3}}, \quad \eta^2 = \frac{2}{3\xi}, \quad \eta^3 = \frac{C_2}{\xi^2}, \quad \eta^4 = \frac{C_2}{18 \xi^4}.$$

The remaining system consists of only one equation:

$$h_t + \frac{C_1 y}{t^{2/3}} h_x + \frac{2y}{3t} h_y = -\frac{2h}{3t}$$

with the general solution

$$h = \frac{1}{y} g\left(\frac{y}{t^{2/3}}, C_1 y t^{1/3} - x\right),$$

where g denotes an arbitrary differentiable two-argument function. Moreover we obtain

$$u = \frac{C_1 y}{t^{2/3}}, \quad v = \frac{2y}{3t}, \quad \varrho = \frac{C_2 y^2}{t^2},$$

$$p = \frac{C_1 y^4}{18 t^4} - \frac{R_H}{2 y^2} g^2\left(\frac{y}{t^{2/3}}, C_1 y t^{1/3} - x\right).$$

These solutions are proper partially invariant if the expressions

$$z_1 \frac{\partial g(z_1, z_2)}{\partial z_2},$$

$$\frac{z_2}{z_1} \frac{\partial g(z_1, z_2)}{\partial z_2} + \frac{\partial g(z_1, z_2)}{\partial z_1} - 2 \frac{g(z_1, z_2)}{z_1}, \quad \frac{1}{z_2^2}$$

with $z_1 = y/t^{2/3}$, $z_2 = C_1 y t^{1/3} - x$ are linearly independent.

7. Concluding Remarks

We have shown that the concept of partially invariant solutions is suitable for the problem of finding exact solutions of a system of partial differential equations of a higher degree of complexity. All solutions were found without considering any boundary or initial conditions. Therefore one has to look for boundary conditions which are satisfied by the given solutions, but this effort is, in general, not very successful in treating real physical problems. At least there is one special advantage of the partially invariant solutions compared with the corresponding invariant solutions: Where the latter contain only free constants, the former depend on arbitrary functions. This allows in certain cases an adaption to a larger variety of boundary and/or initial conditions. Basically, the variety of such conditions allowed by partially invariant solutions depends on their defect δ. Therefore, it seems obvious to start the search for solutions of a special problem with a high defect and then proceed systematically to smaller defects. Such an approach is in progress, the results will be published in another paper.

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